Unidirectional Spectral Singularities

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We propose a class of spectral singularities emerging from the coincidence of two independent singularities with highly directional responses. These spectral singularities result from resonance trapping induced by the interplay between parity-time symmetry and Fano resonances. At these singularities, while the system is reciprocal in terms of a finite transmission, a simultaneous infinite reflection from one side and zero reflection from the opposite side can be realized.

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Among complex potentials, parity-time ($\mathcal{PT}$) symmetry potentials are attractive because they enable real spectra [1]. Such complex potentials have been proposed in optics, electronics, and recently acoustics [2–4]. Depending on the degree of non-Hermiticity, $\mathcal{PT}$ symmetric systems might encounter a phase transition from the exact phase with a real spectrum to the broken phase with a complex spectrum. The transition point, also named the exceptional point, is a topological singularity where the Hamiltonian of the corresponding system becomes defective and the eigenvalues and their associated eigenstates coalesce. While direct physical identification of the exceptional points is difficult, their strong influence on the dynamics can be observed [3–5]. Exceptional points are not the only singularities existing in complex potentials. Another type of singularities are spectral singularities related to the completeness of the continuous spectrum and can satisfy outgoing boundary conditions [6]. Within the scattering matrix formalism, such singularities identify the lasing threshold of cavities with gain [7,8]. Recently, the notion of such spectral singularities extended to the semi-infinite lattices [9], nonlinear potentials [10], and nonreciprocal cavities in the presence of magnetic elements [11]. In the latest one, the presence of a gyrotropic element together with the broken inversion symmetry results in directional lasing.

In this Letter, using the scattering formalism, we introduce a new class of spectral singularities (SS) with a directional response emerging from the interplay of $\mathcal{PT}$ symmetry and Fano resonances. We show that, without breaking the reciprocity, one is able to obtain a simultaneous unidirectional lasing and unidirectional reflectionless mode. For such a mode, reflection in one side of the system tends to infinity, reflection in the other side becomes zero, and the transmission coefficient remains finite. These singularities emerge from the resonance trapping and delay time associated with the reflected signal residing in the gain or loss part of the parity-time symmetric cavity. Remarkably, while always the pseudounitary conservation relation associated with the $\mathcal{PT}$ systems is satisfied, the amplitude of the transmission coefficient is sensitive to the path we take in the plane of gain-loss parameter γ and wave vector q to approach the singular point. In addition, in the absence of loss (gain) and at threshold gain (loss), the structure still acts as a unidirectional laser (reflectionless system). In the passive-loss case, our structure acts as a unidirectional perfect absorber. When the system possesses pure balanced gain, transmission and reflection from the left and right sides of the system tend to infinity and we recover the conventional lasing modes.

In order to demonstrate the SS modes with a directional response, as depicted schematically in Fig. 1, we consider a one-dimensional (1D) chain of evanescently coupled microcavities with resonance frequency ω_c. Without loss of generality, we set all the couplings in the chain to one. In the middle of the chain, we embed a $\mathcal{PT}$ symmetric defect with a coupling strength κ to the chain. The defect is composed of two coupled microcavities with resonance frequency ω_c, in which one possesses gain and the other experiences loss. The coupling strength between the balanced gain and loss microcavities is denoted as κ_c. Finally, we introduce a passive cavity with resonance frequency $\omega_0 + \omega_c$ and couple it to both the gain and loss cavities with a coupling strength c. Interaction between the cavity chain including the $\mathcal{PT}$ dimer, corresponding to a continuum, and the passive disk coupled to the $\mathcal{PT}$ dimer, serving as a discrete state, results in Fano resonances. At the Fano resonance frequency, photons can take different paths to exit from the scattering region. An obvious choice for the photons is a direct path through the $\mathcal{PT}$ dimer, while another option is an indirect path through the passive cavity. Interference between the photons taking different paths leads to a delay in the *time of flight* of the photons and a mechanism to trap them. A delicate design of the interference process results in a tremendous delay for the reflected photons and, consequently, causes amplification or annihilation of photons residing in the gain or loss side, respectively.

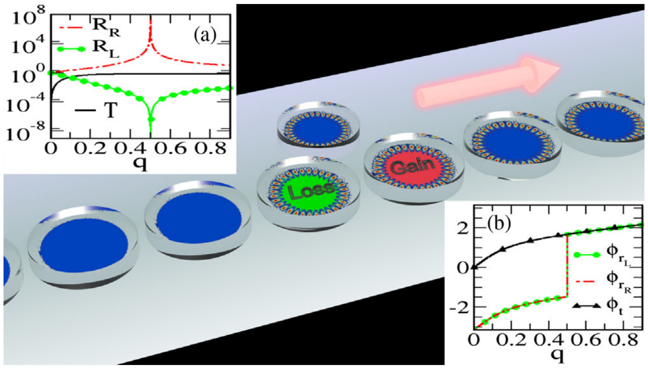

FIG. 1 (color online). Main panel: Schematic of the $\mathcal{PT}$ symmetric Fano coupled disk resonators in Eq. (1). The gain and loss disks are coupled to the passive microdisk with resonance frequency ω_0 and are embedded in the chain of passive disk resonators. At the threshold value of the gain and loss, the reflection from the gain side (pink arrow) diverges and the reflection from the loss side tends to zero resulting in a RLLR mode. (a) Reflections, transmission coefficient, and (b) phase of the transmission and reflection amplitudes for $q^\star = 0.5$, $\kappa = 2.1$, $c = 1$, and $\gamma = \gamma_{th} \simeq 2.11$ versus the wave vector q. The abrupt phase change at the singularity results in the resonance trapping and large delay time for the photons being reflected from the structure. This resonance trapping annihilates the reflection of the system from the left (loss side) and amplifies it from the right (gain side) and causes it to diverge.

In our system, each disk supports two degenerate modes: a clockwise a^+ and a counterclockwise a^-. Using the coupled mode theory, we express the dynamics of the total field amplitudes $\phi = a^+ + a^-$ in each disk [12,13]:

$$i\frac{d\phi_n}{dt} = -\delta_{n\pm1,\pm2}\phi_{n\pm1} - \delta_{n,n\pm1}\kappa\phi_\pm + \omega_c\phi_n \quad (n = \pm1),$$
$$i\frac{d\phi_n}{dt} = -\phi_{n-1} - \phi_{n+1} + \omega_c\phi_n \quad (|n| > 2),$$
$$i\frac{d\phi_0}{dt} = -c(\phi_+ + \phi_-) + (\omega_0 + \omega_c)\phi_0,$$
$$i\frac{d\phi_\pm}{dt} = -\kappa\phi_{\pm1} - \kappa_c\phi_\mp - c\phi_0 \pm i\gamma\phi_\pm + \omega_c\phi_\pm. \tag{1}$$

Here,

$$\delta_{i,j} = \begin{cases} 1 & i = j, \\ 0 & i \neq j \end{cases}$$

is the Kronecker delta and ϕ_n, ϕ_0, and $\phi_\pm$ are the total field amplitude at the disk n, passive cavity, and gain (loss) cavity with subindex $+$ $(-)$, respectively. The coupling terms κ, κ_c, and c are normalized with respect to the coupling strength in the chain. Equations (1) indicate that our structure is a $\mathcal{PT}$ symmetric system, as it is invariant under combined parity operation $\pm n \to \mp n$, $\pm \to \mp$ and time reversal operation $i \to -i$. The chain supports the dispersion relation $\omega = \omega_c - 2\cos(q)$, with $-\pi \le q \le \pi$.

In the elastic scattering process for which $\phi = \psi e^{-i\omega t}$, the stationary modal amplitudes of the system have the asymptotic behavior $\psi_n = F_L e^{iq(n+1)} + B_L e^{-iq(n+1)}$ for $n \le -1$ and $\psi_n = F_R e^{iq(n-1)} + B_R e^{-iq(n-1)}$ for $n \ge 1$, respectively. The amplitude of the forward $F_{L,R}$ and backward $B_{L,R}$ propagating waves in the chain are related by a 2×2 transfer matrix

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

such that

$$\begin{pmatrix} F_R \\ B_R \end{pmatrix} = M \begin{pmatrix} F_L \\ B_L \end{pmatrix}.$$

Elements of the transfer matrix are related to the transmission and reflection amplitudes for the left (L) and right (R) incident waves through the following relations:

$$t_L = t_R \equiv t = \tfrac{1}{M_{22}}, \qquad r_L = -\tfrac{M_{21}}{M_{22}}\, r_R = \tfrac{M_{12}}{M_{22}}. \tag{2}$$

In the framework of the semiclassical laser theory, a self-oscillating laser without any injection satisfies the boundary conditions $F_L = B_R = 0$, $F_R \neq 0$, $B_L \neq 0$. These boundary conditions are mathematically represented by the Jost solutions $\psi_n(q\pm) \to e^{\pm iqn}$ as $n \to \pm\infty$ [6]. Imposing these boundary conditions on Eq. (2), we obtain a *laser* solution indicated with transmission and reflection amplitude tending to infinity at wave vector $q^\star \in \Re$ if and only if $M_{22}(q^\star) = 0$ [6]. The wave vector $q^\star$ that satisfies this relation is the "conventional" SS or lasing mode. However, in order to realize a *unidirectional lasing* mode, in addition to the incoming waves, one of the outgoing waves should be zero:

$$F_L = B_R = F_{R(L)} = 0, \qquad B_{L(R)} \neq 0. \tag{3}$$

Equation (3) is associated with the left (right) lasing mode and solutions $\psi_n(q-) \to e^{-iqn}$, $\psi_n(q+) \to 0$ [$\psi_n(q-) \to 0$, $\psi_n(q+) \to e^{iqn}$] as $n \to \pm\infty$. In general, satisfying conditions in Eq. (3) by using the conventional lasing mode is drastically difficult, as one needs to break the reciprocity for the lasing modes [14]. We recall that breaking reciprocity in optics is a challenging problem by itself. Moreover, introducing gain into the nonreciprocal structures with different transmissions from the different channels does not essentially lead to unidirectional lasing, as it needs divergence of the transmission coefficient of only one side and, at the same time, the convergence (finite) reflection coefficient of the same side.

Surprisingly, we can satisfy the boundary conditions described by Eq. (3) for the left (right) lasing by using Eqs. (2) with $M_{21(12)}(q^\star) \to \infty$ and $M_{22}(q^\star) \neq 0$. In this case, the left (right) reflection coefficients diverge to infinity for $q^\star \in \Re$. More interestingly, we consider a more severe boundary condition when the left (right) reflection tends to infinity and the reflection from the right (left) approaches zero, which is feasible with

$$M_{21(12)}(q^\star) \to \infty, \qquad M_{12(21)}(q^\star) \to 0, \qquad M_{22}(q^\star) \neq 0. \tag{4}$$

Equation (4) results in a very specific singularity where there is a simultaneous left (right) unidirectional lasing and right (left) reflectionless mode. Note that, as long as $M_{22}(q^\star)$ is a finite number, the transmission remains finite. In the following, as depicted in Fig. 1(a), we show that one can satisfy such a relation in the presence of $\mathcal{PT}$ symmetry and Fano resonances.

The elements of the transfer matrix M associated with the system described by Eq. (1) can be expressed as

$$M_{22} = \frac{-Y(\gamma,q)}{2\Gamma(q)\kappa^2\sin(q)}, \qquad M_{21(12)} = \frac{(-)X_{L(R)}(\gamma,q)}{2\Gamma(q)\kappa^2\sin(q)}, \tag{5}$$

where $\Gamma(q) \equiv c^2/(\omega-\omega_c-\omega_0) - \kappa_c$ and

$$Y(\gamma,q) = e^{-2iq}[\Gamma^2 - (\alpha-\xi_+)(\alpha-\xi_-)],$$
$$X_{L(R)}(\gamma,q) = (\alpha^* - \xi_{-(+)})(\alpha-\xi_{+(-)}) - \Gamma^2, \tag{6}$$

with $\xi_\pm \equiv \omega_c + (c^2/(\omega-\omega_c-\omega_0)) \pm i\gamma$ and $\alpha \equiv \kappa^2 e^{iq} + \omega$. In our $\mathcal{PT}$ symmetric setup, we find $M_{11}(q) = M_{22}^*(q)$ for $q \in \Re$ by using Eq. (1) where $*$ stands for conjugation [15].

The conditions given in Eqs. (4) for right lasing-left reflectionless (RLLR) are satisfied if

$$\lim_{(\gamma,q)\to(\gamma_{th},q^\star)} Y \to 0, \qquad \lim_{(\gamma,q)\to(\gamma_{th},q^\star)} X_L \to 0,$$
$$\lim_{(\gamma,q)\to(\gamma_{th},q^\star)} X_R \neq 0, \qquad \lim_{q\to q^\star}\Gamma \to 0. \tag{7}$$

At the SS $q = q^\star (\omega = \omega^\star)$ where we have RLLR, from $\lim_{q\to q^\star}\Gamma \to 0$ we can deduce that $\kappa_c^\star = c^2/(\omega^\star - \omega_c - \omega_0)$. In addition, this relation together with the condition for zero reflection from the left side of the system, namely, $\lim_{(\gamma,q)\to(\gamma_{th},q^\star)} X_L \to 0$ at the SS, leads to $\gamma \to \gamma_{th} = \kappa^2\sin(q^\star)$ and $\kappa_c^\star = (\kappa^2-2)\cos(q^\star)$. The critical coupling $\kappa_c^\star$ yields the critical resonance frequency of the passive defect $\omega_0^\star = -2cos(q^\star) - c^2/\kappa_c^\star$. Notice that these conditions also lead to $\lim_{(\gamma,q)\to\gamma_{th},q^\star} Y \to 0$.

To understand the physical behavior of SS with $\kappa_c^\star$ and $\omega_0^\star$, we expand Eqs. (6) to the first order of $\delta q \equiv q - q^\star$ and $\delta\gamma \equiv \gamma - \gamma_{th}$:

$$X_L \cong -2\kappa^2\cos(q^\star)\delta\gamma\delta q, \tag{8a}$$

$$X_R \cong 2\kappa^2\cos(q^\star)(2\gamma_{th}+\delta\gamma)\delta q + 4\gamma_{th}(\gamma_{th}+\delta\gamma), \tag{8b}$$

$$Y \cong -\beta\delta q - 2e^{-2iq^\star}\gamma_{th}(1-2i\delta q)\delta\gamma, \tag{8c}$$

$$\Gamma \cong -2\left(\frac{\kappa_c^\star}{c}\right)^2 \sin(q^\star)\delta q. \tag{8d}$$

Here, β is a function of $q^\star$ with a finite value. It is noted that when γ approaches to its threshold value, viz. $\delta\gamma \to 0$, the expressions X_L, X_R, and Y approach zero, $4[\gamma_{th}{}^2 + \kappa^2\cos(q^\star)\gamma_{th}\delta q]$, and $-\beta\delta q$, respectively. The latest together with Eqs. (2), (5), and (8d) reveals the fact that the transmission amplitude remains finite at the limit $\delta q \to 0$. Note that in these sequence of limits, namely, $(\delta\gamma, \delta q) \to 0$, X_L approaches zero faster than Y and Γ. Therefore, the left reflection $r_L = X_L/Y$ is zero, and the system becomes unidirectional reflectionless. In contrast, right reflection $r_R = X_R/Y$ tends to infinity as X_R remains finite, and our $\mathcal{PT}$ symmetric setup becomes a unidirectional laser. Moreover, transmission amplitude $t = 1/M_{22} \propto \Gamma/Y$ approaches $t_1 \equiv -4\gamma_{th}{}^2\kappa_c^{\star 2}/\beta\kappa^2c^2$. Figure 1(a) shows the reflection and transmission coefficients for $q^\star = 0.5$, $\kappa = 2.1$, and $c = 1$ versus the wave vector q where we encounter a RLLR mode at $q^\star$. We have checked that our RLLR singularity satisfy the pseudounitary conservation relation $\sqrt{R_L R_R} = |T-1|$ for 1D $\mathcal{PT}$ symmetric systems [16], where $R_{L(R)} \equiv |r_{L(R)}|^2$ and $T \equiv |t|^2$. In our $\mathcal{PT}$ setup, although one side of the reflection is zero, the transmission might not be unity. This means that a RLLR singularity might not be an anisotropic transmission resonance defined in Ref. [16], in which the transmission becomes unity and one side of the reflection approaches zero. As a result, in the RLLR singularity, the system from the lasing side is superunitary (some flux gained) and from the reflectionless side might be subunitary (some flux lost).

The underlying physical mechanism of a RLLR singularity is the Fano resonance trapping and coincidence of different singularities. When resonance trapping occurs for the transmission or reflection, the corresponding delay time, which is proportional to the time that a wave spends inside the potential before it exits from the scattering region, diverges. The delay time is defined as $\tau_{t,r} \equiv d\theta_{t,r}/dq$, where $\theta_{t,r}$ is the argument of the transmission or reflection coefficient [17,18]. In Fig. 1(b), we plotted the phase of the transmission and reflections versus the wave vector. At the wave vector $q = q^\star$ associated with the singularity, there is an abrupt phase change in the reflections, which is an indication of divergence of the delay time for the reflections. Intuitively, light reflecting from the right side will delay in the gain microdisk for a long time and becomes enhanced, while light reflecting from the left side would remain in the lossy microdisk until it becomes completely absorbed. This implies that we have a coincidence of two singularities, one with an amplifying zero width resonance and the other with an annihilating feature. Creation of a unidirectional reflectionless singularity via the resonance trapping is common in non-Hermitian systems and more specifically in $\mathcal{PT}$ symmetric systems (see, for example, [10,16,19]) as in such a case the

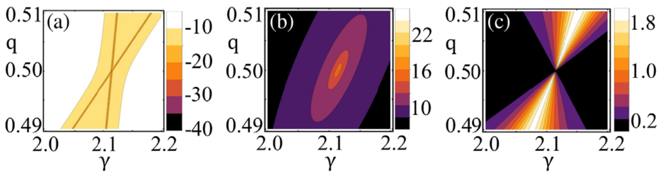

FIG. 2 (color online). Density plot of (a) the logarithm of left reflection, (b) the logarithm of right reflection and (c) the transmission coefficient. At the SS $(\gamma, q) = (\gamma_{th}, q^\star) \simeq (2.1, 0.5)$, the left reflection tends to zero, the right reflection coefficient tends to infinity, and the transmission is a multivalued function. The value of transmission depends on the path one takes in the plane of (γ, q) to approach the singularity. The couplings are the same as the ones used in Fig. 1.

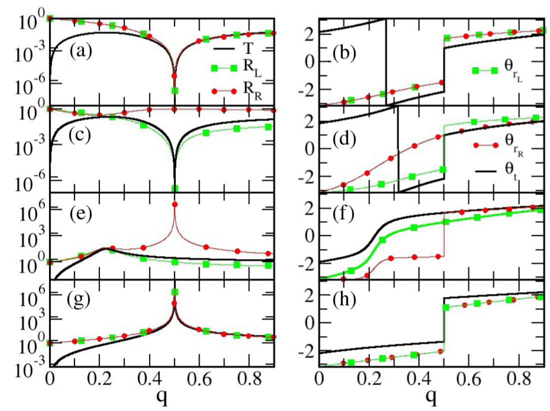

FIG. 3 (color online). Scattering characteristics of Fano model in the schematic of Fig. 1 without $\mathcal{PT}$ symmetry when we have (a) a loss-loss dimer depicting a perfect absorber, (c) a loss-passive depicting a unidirectional perfect absorber, (e) passive-gain results in a finite left reflection and transmission, while the reflection from the right (gain) side tends to infinity indicating a unidirectional laser, (g) gain-gain recovering the conventional SS where the cavity lases from both the left and right sides. (b),(d), (f),(h) Phases of the transmission and reflections of (a),(c),(e),(g), respectively. In all panels, the coupling values are the same as the ones used in Fig. 1.

reflection is bounded from above and smaller than one. On the other hand, to form a unidirectional lasing mode in 1D open systems, one needs to design the system such that the delay time associated with the photons reflecting from one side of the system diverge and, as a result, reflection becomes unbounded. In this respect, utilizing Fano resonances offers an elegant way to trap the light and enlarge the time delay.

Resonances are sensitive to the geometry of the scattering region and its coupling to the environment through the open channel. In the $\mathcal{PT}$ symmetric systems, another parameter that affects the resonances is the gain and loss parameter which enriches the dynamics of $\mathcal{PT}$ symmetric systems. While we always have RLLR at the singularity, without violating the pseudounitary conservation relation mentioned above, the transmission coefficient will be affected by the *path* we take in the plane of the gain or loss parameter and wave vector to approach the SS with a directional response. This implies that, at the singularity, the transmission is a multivalued function and its phase and magnitude depend on the path that we take to approach it. For example, if we interchange the sequence of the limits that we took previously, namely, $(\delta\gamma, \delta q) \to 0$, in order to calculate the transmission amplitude, i.e., $(\delta q, \delta\gamma) \to 0$, we obtain a different transmission amplitude than t_1. In other words, in this new sequence of approaching the SS from Eqs. (8c)) and 8(d), we immediately see that $\lim_{\delta q \to 0} Y = -2e^{-2iq^\star}\gamma_{th}\delta\gamma$, $\lim_{\delta q\to 0}\Gamma = 0$. As a result, even after taking the limit $\delta\gamma \to 0$, the transmission amplitude would be zero. This indicates that, by tuning the wave vector and turning on the gain and loss parameter, at the singularity our setup is from one side a *perfect absorber* and from the other side a laser. Figure 2 presents a typical density plot of the transmission and reflection coefficients versus the wave vector and gain or loss parameter. As mentioned before, the reflections remain intact when we approach the singularity from different paths, while the transmission coefficient is clearly not a unique number, as the singularity may be approached with a transmission obtaining any value between $T = 0$ and $T = 1.8$.

As the RLLR singularity is the coincidence of two singularities, it would be interesting to see the effect of absence of $\mathcal{PT}$ symmetry such that the gain and loss are not exactly balanced and only one of them reaches the threshold. As the main mechanism to create a SS is the resonance trapping, at the threshold of loss in one side we always have a fully vanishing reflection amplitude from the loss side irrespective of the value of the gain (including zero gain) on the other side. The same goes for the gain side; namely, at the threshold of gain, irrespective of the value of loss on the other side, the reflection from the gain side tends to infinity. In the case of both sides having the same amount of gain, we will recover the conventional lasing singularity in which $M_{22}(q^\star) = 0$ and $M_{12(21)}(q^\star)$ is nonzero. Finally, when we have a loss-loss scenario at the threshold, the same singularity occurs for left and right sides. In this case, both reflections and transmission amplitudes vanish, and the setup becomes a perfect absorber. Figure 3 summarizes these results.

Experimentally, the coupled cavity array scenario can be realized by other forms such as microring optical cavity [20] or photonic crystal structures [21]. The optical gain and loss, for example, can be achieved with InGaAsP quantum wells and chrome layers on top of the cavity, respectively [22]. In the system, by including the intrinsic loss of the passive cavities, it can be found that, if all the passive cavities possess the same intrinsic loss, the unidirectional SS are maintained with a shifted gain or loss threshold value. This reveals that the SS are robust in the presence of intrinsic cavity decays.

We have shown that the interplay of $\mathcal{PT}$ symmetry and Fano resonances can result in a coincidence of two SS with unique behaviors. One singularity creates a subunitary flux dynamics from one side with a vanishing reflection and the other with zero width resonance resulting in infinite reflection from the opposite side of the setup. Furthermore, we discussed the case where gain and loss are not balanced. We found that at the threshold of gain or loss one can observe the divergence of reflection or zero reflection, respectively, on the side that reaches the threshold value. Our study brings a new class of singularities that has not been considered before and at the same time opens a new direction to design new laser cavities with extra freedom. Of interest will be to investigate how these singularities can affect the dynamics in the presence of nonlinearity where nonlinear Fano resonances affect the transmission [23].

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[1] C. M. Bender and S. Boettcher, Phys. Rev. Lett. **80**, 5243 (1998).
[2] A. Ruschhaupt, F. Delgado, and J. G. Muga, J. Phys. A **38**, L171 (2005).
[3] H. Ramezani, J. Schindler, F. M. Ellis, U. Gunther, and T. Kottos, Phys. Rev. A **85**, 062122 (2012).
[4] X. Zhu, H. Ramezani, C. Shi, J. Zhu, and X. Zhang, Phys. Rev. X **4**, 031042 (2014).
[5] C. Dembowski, H.-D. Gräf, H. L. Harney, A. Heine, W. D. Heiss, H. Rehfeld, and A. Richter, Phys. Rev. Lett. **86**, 787 (2001); C. Dembowski, B. Dietz, H.-D. Gräf, H. L. Harney, A. Heine, W. D. Heiss, and A. Richter, Phys. Rev. E **69**, 056216 (2004); J. Schindler, Z. Lin, J. M. Lee, H. Ramezani, F. M. Ellis, and T. Kottos, J. Phys. A **45**, 444029 (2012).
[6] A. Mostafazadeh, Phys. Rev. Lett. **102**, 220402 (2009).
[7] K. M. Farham, H. Schomerus, M. Patra, and C. W. J Beenakker, Europhys. Lett. **49**, 48 (2000); M. Chitsazi, S. Factor, J. Schindler, H. Ramezani, F. M. Ellis, and T. Kottos, Phys. Rev. A **89**, 043842 (2014); M. Brandstetter, M. Liertzer, C. Deutsch, P. Klang, J. Schöberl, H. E. Türeci, G. Strasser, K. Unterrainer, and S. Rotter, Nat. Commun. **5**, 4034 (2014).
[8] For a review on scattering in the 1D non-Hermitian potential, see J. G. Muga, J. P. Palaob, B. Navarroa, and I. L. Egusquiza, Phys. Rep. **395**, 357 (2004).
[9] S. Longhi, Phys. Rev. B **80**, 165125 (2009).
[10] A. Mostafazadeh, Phys. Rev. Lett. **110**, 260402 (2013); Phys. Rev. A **87**, 063838 (2013).
[11] H. Ramezani, S. Kalish, I. Vitebskiy, and T. Kottos, Phys. Rev. Lett. **112**, 043904 (2014).
[12] M. F. Yanik and S. Fan, Phys. Rev. Lett. **92**, 083901 (2004); H. Ramezani, T. Kottos, V. Shuvayev, and L. Deych, Phys. Rev. A **83**, 053839 (2011).
[13] One can write the same set of equations for the subtraction of the clockwise and counterclockwise modes where all the couplings change their sign and the dispersion relation becomes $\omega = \omega_c + 2\cos(q)$. We carried out the same analysis and obtained the same results.
[14] In nonreciprocal systems, left and right transmission coefficients are not equal and one needs to modify Eq. (2) [24].
[15] S. Longhi, Phys. Rev. A **82**, 031801(R) (2010).
[16] Li Ge, Y. D. Chong, and A. D. Stone, Phys. Rev. A **85**, 023802 (2012).
[17] R. Landauer and T. Martin, Rev. Mod. Phys. **66**, 217 (1994).
[18] E. H. Hauge, J. P. Falck, and T. A. Fjeldly, Phys. Rev. B **36**, 4203 (1987).
[19] Z. Lin, H. Ramezani, T. Eichelkraut, T. Kottos, H. Cao, and D. N. Christodoulides, Phys. Rev. Lett. **106**, 213901 (2011).
[20] F. N. Xia, L. Sekaric, and Y. Vlasov, Nat. Photonics **1**, 65 (2007).
[21] M. Notomi, E. Kuramochi, and T. Tanabe, Nat. Photonics **2**, 741 (2008).
[22] L. Feng, Zi Jing Wong, R. Ma, Y. Wang, and X. Zhang, Science **346**, 972 (2014).
[23] A. E. Miroshnichenko, B. A. Malomed, and Y. S. Kivshar, Phys. Rev. A **84**, 012123 (2011); F. Nazari, N. Bender, H. Ramezani, M. K. Moravvej-Farshi, D. N. Chirstodoulides, and T. Kottos, Opt. Express **22**, 9574 (2014).
[24] H. Ramezani, Z. Lin, S. Kalish, T. Kottos, V. Kovanis, and I. Vitebskiy, Opt. Express **20**, 26200 (2012).